Statistical Models and Data Analysis

Summer term 2018

Problem Set 4 / Solutions — 7.5.2018

1. (Orthogonality of sine and cosine functions)

Let us define a "dot product" between two functions $f(t)$ and $g(t)$ as the integral $T^{-1}\int_0^T f(t)g(t)dt$.

(a) Calculate this product for $f(t) = \sin(\frac{2\pi nt}{T})$ and $g(t) = \sin(\frac{2\pi mt}{T})$ where n and m are integer numbers. Perform this calculation without using your knowledge about complex numbers, i.e., use the addition theorems for the sine and cosine. Pay particular attention to the case $n = m$.

(b) Perform the same calculation for $f(t) = \sin(\frac{2\pi nt}{T})$ and $g(t) = \cos(\frac{2\pi mt}{T})$.

(c) Solve these problems again but now represent the sine- and cosine-functions by complex-valued exponential functions as derived in the first exercise. Carry out the integrals as you would do for a real-valued exponential function, i.e., the antiderivative of $\exp(\mathrm{i}ax)$ is

$$\frac{1}{ia}\exp(\mathrm{i}ax).$$

Solution:

(a) Let us first use the substitution $\tau = 2\pi t/T$. (Note $d\tau = 2\pi/T dt$):

$$\frac{1}{T}\int_0^T \sin\left(\frac{2\pi nt}{T}\right)\sin\left(\frac{2\pi mt}{T}\right)dt = \frac{1}{2\pi}\int_0^{2\pi}\sin(n\tau)\sin(m\tau)\,d\tau$$

Recall

$$\begin{aligned}\cos(a+b) &= \cos(a)\cos(b) - \sin(a)\sin(b)\\ \cos(a-b) &= \cos(a)\cos(b) + \sin(a)\sin(b)\\ \sin(a+b) &= \sin(a)\cos(b) + \cos(a)\sin(b)\\ \sin(a-b) &= \sin(a)\cos(b) - \cos(a)\sin(b)\end{aligned}$$

Therefore,

$$\sin(a)\sin(b) = \frac{1}{2}\left(\cos(a-b) - \cos(a+b)\right)$$

And so,

$$\frac{1}{2\pi}\int_0^{2\pi}\sin(n\tau)\sin(m\tau)\,d\tau = \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}\left(\cos((n-m)\tau) - \cos((n+m)\tau)\right)\,d\tau$$

First, take the case $m \neq n$ and $m \neq -n$

$$\frac{1}{2\pi}\int_0^{2\pi}\sin(n\tau)\sin(m\tau)\,d\tau = \frac{1}{4\pi}\left(\frac{1}{n-m}\sin((n-m)\tau) - \frac{1}{n+m}\sin((n+m)\tau)\right)\Bigg|_{\tau=0}^{\tau=2\pi}$$

As $n, m \in Z$ and the sine functions are periodic for integer multiples of 2π,

$$\frac{1}{2\pi}\int_0^{2\pi}\sin(n\tau)\sin(m\tau)\,d\tau = 0 \qquad m \neq n \text{ and } m \neq -n$$

Now treat the case $n = m$

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \sin(n\tau)\sin(m\tau)\,d\tau =&\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}\left(\cos(0\tau)-\cos(2n\tau)\right)\,d\tau\\
=&\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}\left(1-\cos(2n\tau)\right)\,d\tau\\
=&\frac{1}{2\pi}\left(\frac{\tau}{2}-\frac{1}{2n}\sin(2n\tau)\right)\Big|_{\tau=0}^{\tau=2\pi}\\
=&\frac{1}{2}
\end{aligned}$$

Similarly, for $n = -m$,

$$\frac{1}{2\pi}\int_0^{2\pi}\sin(n\tau)\sin(-n\tau)\,d\tau = -\frac{1}{2}$$

Unless, of course, $n = 0$ or $m = 0$, in which case $\sin(n\tau) = \sin(0) = 0$ identically, in which case the integral is zero, too.

(b)

$$\frac{1}{2\pi}\int_0^{2\pi}\sin(n\tau)\cos(m\tau)\,d\tau =\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}\left(\sin((n+m)\tau)+\sin(n-m)\tau\right)d\tau$$

for $m \neq n$ and $m \neq -n$

$$\begin{aligned}
=&\frac{1}{2\pi}\left(-\frac{1}{n+m}\cos((n+m)\tau)-\frac{1}{n-m}\cos(n-m)\tau\right)\Big|_{\tau=0}^{\tau=2\pi}\\
=&0
\end{aligned}$$

on the other hand, if $n = m$ or $n = -m$, then one of the integrands becomes zero.Which implies that

$$\frac{1}{2\pi}\int_0^{2\pi}\sin(n\tau)\cos(m\tau)\,d\tau =0$$

for all $m, n \in Z$.

(c) To make life simpler, assume that $m > 0$ and $n > 0$. Negative m or n can always be subsumed, as $\cos(-n\tau) = \cos(n\tau)$ and $\sin(-m\tau) = -\sin(m\tau)$.

Use the fact that

$$\exp(in\tau) = \cos(n\tau) + \mathrm{i}\sin(n\tau)$$

to write

$$\frac{1}{2\pi}\int_0^{2\pi}\exp(\mathrm{i}m\tau)\exp(\mathrm{i}n\tau)\ d\tau =\frac{1}{2\pi}\int_0^{2\pi}(\cos(n\tau)\cos(m\tau)-\sin(n\tau)\sin(m\tau))+\mathrm{i}\left(\cos(n\tau)\sin(m\tau)+\sin(n\tau)\cos(m\tau)\right)d\tau \quad (1)$$

$$\frac{1}{2\pi}\int_0^{2\pi}\exp(\mathrm{i}m\tau)\exp(-\mathrm{i}n\tau)\ d\tau =\frac{1}{2\pi}\int_0^{2\pi}(\cos(n\tau)\cos(m\tau)+\sin(n\tau)\sin(m\tau))+\mathrm{i}\left(\cos(n\tau)\sin(m\tau)-\sin(n\tau)\cos(m\tau)\right)d\tau \quad (2)$$

On the other hand,

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi}\exp(\mathrm{i}m\tau)\exp(\mathrm{i}n\tau)\ d\tau =&\frac{1}{2\pi}\int_0^{2\pi}\exp(\mathrm{i}(m+n)\tau)\ d\tau\\
=&\frac{1}{\mathrm{i}(m+n)}\exp(\mathrm{i}(m+n)\tau)\Big|_{\tau=0}^{\tau=2\pi}.
\end{aligned}$$

Therefore, given that $m + n$ is integer and $\exp(2\pi \mathrm{i}k) = 1$ for $k \in \mathbb{Z}$,

$$\frac{1}{2\pi}\int_0^{2\pi}\exp(im\tau)\exp(in\tau)\ d\tau =0$$

Likewise,

$$\frac{1}{2\pi}\int_0^{2\pi}\exp(\mathrm{i}m\tau)\exp(-\mathrm{i}n\tau)\ d\tau = \frac{1}{2\pi}\int_0^{2\pi}\exp(\mathrm{i}(m-n)\tau)\ d\tau = \begin{cases}0 & m \neq n\\ 1 & m = n\end{cases}$$

Then we have

$$\frac{\text{Eq. } 1 + \text{Eq.}2}{2} = \frac{1}{2\pi}\int_0^{2\pi}\cos(n\tau)\cos(m\tau) + \mathrm{i}\cos(n\tau)\sin(m\tau)\ d\tau = 0 \qquad n \neq m$$

Both the real and imaginary parts must integrate to zero. Therefore,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\cos(n\tau)\cos(m\tau)\ d\tau =& 0 \qquad n \neq m\\ \frac{1}{2\pi}\int_0^{2\pi}\cos(n\tau)\sin(m\tau)\ d\tau =& 0\end{aligned}$$

Likewise,

$$\frac{\text{Eq. } 2 - \text{Eq.}1}{2} = \frac{1}{2\pi}\int_0^{2\pi}\sin(n\tau)\sin(m\tau) - \mathrm{i}\sin(n\tau)\cos(m\tau)\ d\tau = 0 \qquad n \neq m$$

and thus

$$\frac{1}{2\pi}\int_0^{2\pi}\sin(n\tau)\sin(m\tau)\ d\tau = 0 \qquad n \neq m.$$

For $n = m$ we have

$$\frac{1}{2\pi}\int_0^{2\pi}\sin(n\tau)^2\ d\tau = \frac{1}{2\pi}\int_0^{2\pi}\cos(n\tau)^2\ d\tau = \frac{1}{2}$$

Alternatively, write

$$\begin{aligned}\sin(n\tau) =& \frac{1}{2i}\left(\exp(in\tau) - \exp(-in\tau)\right)\\ \cos(m\tau) =& \frac{1}{2}\left(\exp(in\tau) + \exp(-in\tau)\right)\end{aligned}$$

Hence,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\sin(n\tau)\sin(m\tau)\,d\tau =& \frac{1}{2\pi}\left(-\frac{1}{4}\right)\int_0^{2\pi}\left(\exp(in\tau) - \exp(-in\tau)\right)\left(\exp(im\tau) - \exp(-im\tau)\right)\ d\tau\\ =& -\frac{1}{8\pi}\int_0^{2\pi}\left(\exp(i(n+m)\tau) + \exp(-i(n+m)\tau) - \exp(i(m-n)\tau) - \exp(i(n-m)\tau)\right)\ d\tau\end{aligned}$$

and proceed from here to integrate the complex exponentials.

2. (Discrete Fourier Transform)

In class, Fourier transform was introduced by sine and cosine series

$$f(t) = \lim_{n\to\infty}\left[\sum_{k=0}^{n} a_k\ \cos(2\pi kt/T) + \sum_{k=0}^{n} b_k\ \sin(2\pi kt/T)\right]\ . \tag{3}$$

Derive complex-valued constants $\hat{f}_k$ as functions of a_k and b_k such that this series can be expressed as

$$f(t) = \frac{1}{T}\lim_{n\to\infty}\sum_{k=-n}^{n}\hat{f}_k\, \mathrm{e}^{2\pi \mathrm{i}kt/T} \tag{4}$$

To do so, use $\mathrm{e}^{ix} = \cos(x) + \mathrm{i}\ \sin(x)$!

Solution:

$$\sum_{k=0}^{n} a_k \cos\left(\frac{2\pi kt}{T}\right) = \sum_{k=0}^{n} a_k \frac{1}{2}\left[\exp\left(i\frac{2\pi kt}{T}\right) + \exp\left(-i\frac{2\pi kt}{T}\right)\right]$$
$$\sum_{k=0}^{n} b_k \sin\left(\frac{2\pi kt}{T}\right) = \sum_{k=0}^{n} b_k \frac{1}{2i}\left[\exp\left(i\frac{2\pi kt}{T}\right) - \exp\left(-i\frac{2\pi kt}{T}\right)\right]$$

Let us take $k > 0$, If we add the two equations above and compare to

$$\frac{1}{T}\sum_{k=-n}^{n} \hat{f}_k \, \mathrm{e}^{2\pi \mathrm{i}kt/T},$$

we get

$$\frac{1}{T}\hat{f}_k = \frac{1}{2}(a_k - ib_k) \qquad k > 0$$

For negative k,we have

$$\frac{1}{T}\hat{f}_{-k} = \frac{1}{2}(a_k + ib_k) \qquad k > 0$$

Lastly, for $k = 0$,

$$\frac{1}{T}f_0 = a_0$$

This last result holds as both exponential terms in the square brackets evaluate to unity for $k = 0$, and hence sum to 2.

Let's start from the opposite approach, using $\mathrm{e}^{ix} = \cos(x) + \mathrm{i}\,\sin(x)$

$$\frac{1}{T}\sum_{k=-n}^{n} \hat{f}_k \, \mathrm{e}^{2\pi \mathrm{i}kt/T} = \frac{1}{T}\hat{f}_0 + \frac{1}{T}\sum_{k=1}^{n}\left(\hat{f}_k \mathrm{e}^{2\pi \mathrm{i}kt/T} + \hat{f}_{-k}\mathrm{e}^{-2\pi \mathrm{i}kt/T}\right)$$
$$= \frac{1}{T}\hat{f}_0 + \frac{1}{T}\sum_{k=1}^{n}\left(\hat{f}_k \cos(2\pi kt/T) + \hat{f}_{-k}\cos(-2\pi kt/T)\right) + i\left(\hat{f}_k \sin(2\pi kt/T) + \hat{f}_{-k}\sin(-2\pi kt/T)\right)$$

Using $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$

$$\frac{1}{T}\sum_{k=-n}^{n} \hat{f}_k \, \mathrm{e}^{2\pi \mathrm{i}kt/T} == \frac{1}{T}\hat{f}_0 + \frac{1}{T}\sum_{k=1}^{n}\left(\hat{f}_k \cos(2\pi kt/T) + \hat{f}_{-k}\cos(2\pi kt/T)\right) + i\left(\hat{f}_k \sin(2\pi kt/T) - \hat{f}_{-k}\sin(2\pi kt/T)\right)$$
$$= \frac{1}{T}\hat{f}_0 + \frac{1}{T}\sum_{k=1}^{n}\left(\left(\hat{f}_k + \hat{f}_{-k}\right)\cos(2\pi kt/T)\right) + \left(i\left(\hat{f}_k - \hat{f}_{-k}\right)\sin(2\pi kt/T)\right)$$

Comparing the last line to

$$\sum_{k=0}^{n} a_k \,\cos(2\pi kt/T) + \sum_{k=0}^{n} b_k \,\sin(2\pi kt/T),$$

we identify

$$a_k = \frac{1}{T}\left(\hat{f}_k + \hat{f}_{-k}\right)$$
$$b_k = \frac{1}{T}i\left(\hat{f}_k - \hat{f}_{-k}\right).$$

Multiplying the last line by $-\mathrm{i}$

$$a_k = \frac{1}{T}\left(\hat{f}_k + \hat{f}_{-k}\right)$$
$$-ib_k = \frac{1}{T}\left(\hat{f}_k - \hat{f}_{-k}\right)$$

If we add, we get

$$a_k - ib_k = \frac{1}{T} 2 \hat{f}_k$$

whereas, if we subtract, we get

$$a_k + ib_k = \frac{1}{T} 2 \hat{f}_{-k}$$

Summarizing, we have

$$\begin{aligned} \frac{1}{T} \hat{f}_k &= \frac{1}{2} \left(a_k - ib_k \right) \\ \frac{1}{T} \hat{f}_{-k} &= \frac{1}{2} \left(a_k + ib_k \right) \end{aligned}$$

and

$$\frac{1}{T} \hat{f}_0 = a_0$$