Statistical Models and Data Analysis

Summer term 2018

Problem Set 3 / Solutions 23.4.2018

1. (Complex numbers)

(a) Compute the real and imaginary part of the following expressions:

$$\mathrm{i} \qquad (3+\mathrm{i})-(5+2\,\mathrm{i}) \qquad (3+\mathrm{i})\,(5+2\,\mathrm{i}) \qquad (3+\mathrm{i})/(5+2\,\mathrm{i}) \qquad (3+\mathrm{i})^2 \qquad 4\,\mathrm{e}^{\mathrm{i}\pi/4}\ .$$

(b) Compute the modulus and argument of the following expressions:

$$\mathrm{i} \qquad 2\,\mathrm{e}^{\mathrm{i}\,\pi/2} \qquad (2+2\,\mathrm{i}) \qquad (\sqrt{3}-\mathrm{i}) \qquad \ln(-2) \qquad \sqrt{-2} \qquad (4\,\mathrm{e}^{\mathrm{i}\pi/4})^5\ .$$

(c) Find all complex numbers z that solve the equations

$$z^2=\mathrm{i} \qquad z^3-1=0 \qquad z^4+2\,z^2+1=0\ .$$

Solution:

(a)

$$z=(3+i)-(5+2i)=-2-i$$
$$z=(3+i)(5+i)=15-1+8i=14+8i$$
$$z=(3+i)/(5+i)=\frac{3+i}{5+i}\,\frac{5-i}{5-i}=\frac{(3+i)(5-i)}{5^2+1^2}=(15+1-3i+5i)\,/26=\frac{16}{26}+i\frac{2}{26}=\frac{8}{13}+\frac{1}{13}i$$
$$(3+i)^2=3^2+6i+i^2=9-1+6i=8+6i$$
$$z=4\,\mathrm{e}^{\mathrm{i}\pi/4}=4\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)=2\sqrt{2}+i\ 2\sqrt{2}$$

(b)

$$i=\exp(i\pi/2);\qquad |i|=1 \qquad \mathsf{arg}(i)=\pi/2$$
$$z=2+2i \qquad |z|=\sqrt{2^2+2^2}=2\sqrt{2} \qquad \mathsf{arg}(z)=\arctan(2/2)=\pi/4$$
$$z=\sqrt{3}-i \qquad |z|=\sqrt{\sqrt{3}^2+1^2}=2 \qquad \mathsf{arg}(z)=\arctan(-1/\sqrt{3})=-\pi/6$$
$$z=\ln(-2)=\ln(2\exp(i\pi))=\ln 2+i\pi \qquad |z|=\sqrt{\pi^2+\ln(2)^2} \qquad \mathsf{arg}(z)=\arctan(\pi/\ln 2)$$
$$z=\sqrt{-2}=\sqrt{2\exp(i\pi))}=\sqrt{2}\exp(i\pi/2)=i\sqrt{2} \qquad |z|=\sqrt{2} \qquad \mathsf{arg}(z)=\pi/2$$
$$z=(4\,\mathrm{e}^{\mathrm{i}\pi/4})^5\ =4^5\exp(i5\pi/4) \qquad |z|=4^5=1024 \qquad \mathsf{arg}(z)=5\pi/4$$

(c) In these questions, we always try to use the polar representation of complex numbers (i.e., modulus + argument of the complex number).

$$z^2=i=\exp(\mathrm{i}\,[\pi/2+2n\pi])$$

true for any $n\in Z$, as for integer n, $\exp(i2n\pi)=1$. Taking the square root

$$z=\exp(\mathrm{i}\,[\pi/4+n\pi])$$

Note that $\exp(i\pi)=-1$. We have two distinct solutions, namely

$$z=\frac{1}{\sqrt{2}}\,(1+i)$$

or

$$z = -\frac{1}{\sqrt{2}}(1+i)$$

depending on whether n is even or odd. For the next problem, we need to find the cube root of unity.

$$z^3 = 1 = \exp(i2n\pi)$$

Distinct solutions are

$$\begin{aligned}
z &= 1 \\
z &= \exp(i2\pi/3) = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\
z &= \exp(i4\pi/3) = \cos(4\pi/3) + i\sin(4\pi/3) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}
\end{aligned}$$

—

The last equation is a quadratic in z^2:

$$z^4 + 2z^2 + 1 = (z^2+1)^2 = 0$$

We have

$$z^2 = -1 = \exp(i(\pi + 2n\pi))$$

with solutions

$$z = \exp(i(\pi/2 + n\pi))$$

Thus, the two distinct solutions are

$$z = i \qquad \text{or} \qquad z = -i$$

2. (Complex functions) Consider $f(z) = z^2$ for $z = x + iy$. Let $u(x,y) = \mathrm{Re}\,[f(z)]$ be the real part and $v(x,y) = \mathrm{Im}\,[f(z)]$ be the imaginary part of $f(z)$. Show that, for this $f(z)$, the following hold:

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}
\end{aligned}$$

The symbols $\partial/\partial x$ and $\partial/\partial y$ symbolize partial derivatives. For instance, $\dfrac{\partial u(x,y)}{\partial y}$ signifies taking the derivative of $u(x,y)$ with respect to y while keeping x constant.

Solution:

(a) We will try to show that this holds generally for analytic $f(z)$.

$$f(z) = f(x + i\,y) = u(x,y) + i\;v(x,y)$$

We have two identities:

$$\begin{aligned}
f'(z)\frac{dz}{dx} = f'(z) &= \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \\
f'(z)\frac{dz}{dy} = if'(z) &= \frac{\partial u}{\partial y} + i\,\frac{\partial v}{\partial y}
\end{aligned}$$

Let's multiply

$$if'(z) = \frac{\partial u}{\partial y} + i\,\frac{\partial v}{\partial y}$$

by $-i$. We obtain

$$f'(z) = -\,i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

which we contrast with

$$f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}$$

Identifying the real parts in both equations and the imaginary parts in both equations, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\,\frac{\partial v}{\partial x}$$

These are known as the Cauchy relations. One reason why analytic (complex) functions are important can be seen by taking further partial derivatives:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$$
$$\frac{\partial^2 u}{\partial y^2} = -\,\frac{\partial^2 v}{\partial x \partial y}$$

Add these two equations together, and we have

$$\nabla^2 u = 0$$

and, likewise, $\nabla^2 v = 0$. Hence u and v solve Poisson's equation $\nabla^2 \phi = 0$.

3. (Complex numbers and matrices)

Consider the real matrices

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

Show that for all matrices $A = x_1\,E + y_1\,I$ and $B = x_2\,E + y_2\,I$, the following matrices C can be expressed as $C = x_3\,E + y_3\,I$ and give the corresponding expressions for x_3 and y_3!

$$C = A + B \qquad C = A\,B \qquad C = B\,A \qquad C = A^{-1}\,B \qquad C = \exp[A]\,\exp[-B]$$

Note that the inverse of a 2x2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ equals

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

as long as $ad - bc \neq 0$.
The exponential of a matrix M is defined as

$$\exp[M] = \sum_{n=0}^{\infty} \frac{M^n}{n!}$$

where $n! = n(n-1)(n-2)\cdots 2\cdot 1$, which is called the factorial of n.

Try to interpret these results in terms of complex numbers.

Solution:

(a) First remark

$$I.I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -E$$

Second (trivial) remark: E is the identity matrix. In other words

$$E.A = A$$

for *any* matrix A. Hence, with $A = x_1\,E + y_1\,I$ and $B = x_2\,E + y_2\,I$, we have

$$C = A + B = (x_1 + x_2)E + (y_1 + y_2)I$$
$$C = AB = (x_1\,E + y_1\,I)(x_2\,E + y_2\,I) = (x_1x_2 - y_1y_2)E + (x_1y_2 + x_2y_1)I$$

Note that, if we write out this product

$$C = \begin{pmatrix} x_1x_2 - y_1y_2 & x_2y_1 + x_1y_2 \\ -x_2y_1 - x_1y_2 & x_1x_2 - y_1y_2 \end{pmatrix}$$

This happens to be symmetric under the interchange of $x_1 \leftrightarrow x_2; y_1 \leftrightarrow y_2$, so

$$C = BA = (x_1x_2 - y_1y_2)E + (x_1y_2 + x_2y_1)I = AB$$

$$C = BA = AB = \begin{pmatrix} \mathrm{Re}(x_1 + \mathrm{i}y_1)(x_2 + \mathrm{i}y_2) & \mathrm{Im}(x_1 + \mathrm{i}y_1)(x_2 + \mathrm{i}y_2) \\ -\mathrm{Im}(x_1 + \mathrm{i}y_1)(x_2 + \mathrm{i}y_2) & \mathrm{Re}(x_1 + \mathrm{i}y_1)(x_2 + \mathrm{i}y_2) \end{pmatrix} \quad (1)$$

Note that, **in general**, for matrices A and B *not* defined as above in terms of E and I, $AB \neq BA$.

(b) Now, for $A = x_1\,E + y_1\,I$,

$$\det A = \det\left[\begin{pmatrix} \mathrm{x1} & \mathrm{y1} \\ -\mathrm{y1} & \mathrm{x1} \end{pmatrix}\right] = x_1^2 + y_1^2$$

and we have

$$A^{-1} = \frac{1}{x_1^2 + y_1^2}\,(x_1E - y_1I)$$

Hence,

$$A^{-1}B = .\frac{1}{x_1^2 + y_1^2}\,((x_1x_2 + y_1y_2)E + (x_1y_2 - x_2y_1)I)$$

(c) This one can be tricky. Either one uses the fact that E and I commute, or one diagonalizes (the more generic method).

First, keep in mind that

$$\exp(A)\exp(-B) = \exp\left\{A - B - \frac{1}{2}[A,B] + \frac{1}{12}\left(-[A,[A,B]] + [B,[B,A]]\right) + \cdots\right\}$$

which is known as the Baker-Campbell-Haussdorff formula, for which we used the shorthand

$$[A,B] = AB - BA,$$

also known as the commutator of A and B. But, as

$$[E,I] = 0,$$

then, if we have $A = x_1\,E + y_1\,I$ and $B = x_2\,E + y_2\,I$,

$$\exp(A)\exp(-B) = \exp(A - B)$$

as all the commutators are zero. So, therefore,

$$\exp(A)\exp(-B) = \exp((x_1 - x_2)\,E + (y_1 - y_2)\,I)$$

Now, apply the rule from eq 1.

$$\exp((x_1 - x_2)\,E + (y_1 - y_2)\,I) = \begin{pmatrix} \mathsf{Re}\exp(x_1 - x_2 + \mathrm{i}(y_1 - y_2)) & \mathsf{Im}\exp(x_1 - x_2 + \mathrm{i}(y_1 - y_2)) \\ -\mathsf{Im}\exp(x_1 - x_2 + \mathrm{i}(y_1 - y_2)) & \mathsf{Re}\exp(x_1 - x_2 + \mathrm{i}(y_1 - y_2)) \end{pmatrix}$$

Using $\exp(x + \mathrm{i}y) = \exp(x)\,[\cos(x) + \mathrm{i}\sin(x)]$,

$$\exp((x_1 - x_2)\,E + (y_1 - y_2)\,I) = \begin{pmatrix} \exp(x_1 - x_2)\cos(y_1 - y_2) & \exp(x_1 - x_2)\sin(y_1 - y_2) \\ \exp(x_1 - x_2)\sin(y_2 - y_1) & \exp(x_1 - x_2)\cos(y_1 - y_2) \end{pmatrix}$$

But this is not a valid approach to solving the general problem. Because it's instructive, let's review the general approach. Start with

$$A = x_1 E + y_1 = \begin{pmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{pmatrix}$$
$$\det(A - \lambda I) = \left| \begin{pmatrix} x_1 - \lambda & y_1 \\ -y_1 & x_1 - \lambda \end{pmatrix} \right| = 0$$

Working out the determinant,

$$\lambda^2 - 2x_1\lambda + x_1^2 + y_1 = 0$$
$$\lambda = \frac{2x_1 \pm \sqrt{4x_1^2 - 4(x_1^2 + y_1^2)}}{2}$$
$$= x_1 \pm i\,y_1$$

Now we can compute the eigenvectors

$$(A - \lambda I)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \mp i y_1 & y_1 \\ -y_1 & \mp i\,y_1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

So that the corresponding eigenvalue/eigenvector combinations

$$\lambda_1 = x_1 + i\,y_1 \qquad v = \begin{pmatrix} i \\ 1 \end{pmatrix}$$
$$\lambda_2 = x_1 - i\,y_1 \qquad v = \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

Define the matrix

$$V = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$$

Note that we haven't normalized the eigenvectors to unit length. We have the inverse

$$V^{-1} = \frac{1}{2i}\begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}.$$

We have

$$V^{-1}AV = \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Now, let's take

$$A^n = VV^{-1}AVV^{-1}AV \cdots V^{-1}AVV^{-1}$$
$$= V\left(V^{-1}AV\right)\cdots\left(V^{-1}AV\right)V^{-1}$$
$$= V\Lambda\cdots\Lambda V^{-1}$$
$$= V\Lambda^n V^{-1}$$
$$= V\begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}V^{-1}$$

In general, if $f(x)$ is a function that has a convergent power series representation, then

$$f(A) = V\begin{pmatrix} f(\lambda_1) & 0 \\ 0 & f(\lambda_2) \end{pmatrix}V^{-1}$$

Let's apply this to the case at hand

$$
\begin{aligned}
\exp(A) &= \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2}\exp(x_1+iy_1) & 0 \\ 0 & \frac{1}{2}\exp(x_1-iy_1) \end{pmatrix} \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix} \\
&= \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2i}\exp(x_1+iy_1) & \frac{1}{2}\exp(x_1+iy_1) \\ -\frac{1}{2i}\exp(x_1-iy_1) & \frac{1}{2}\exp(x_1-iy_1) \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2}\exp(x_1)\left(\exp(iy_1)+\exp(iy_1)\right) & -\frac{1}{2i}\exp(x_1)\left(\exp(iy_1)-\exp(-iy_1)\right) \\ \frac{1}{2i}\exp(x_1)\left(\exp(iy_1)-\exp(-iy_1)\right) & \frac{1}{2}\exp(x_1)\left(\exp(iy_1)+\exp(iy_1)\right) \end{pmatrix} \\
&= \begin{pmatrix} \exp(x_1)\cos(y_1) & -\exp(x_1)\sin(y_1) \\ \exp(x_1)\sin(y_1) & \exp(x_1)\cos(y_1) \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
A &= x_1\, E + y_1\, I \\
A^2 &= \left(x_1^2 - y_1^2\right) E + 2x_1y_1 I \\
A^3 &= \left[x_1\left(x_1^2-y_1^2\right) - 2x_1y_1^2\right] E + \left[2x_1^2y_1 + y_1\left(x_1^2-y_1^2\right)\right] I \\
&= \left(x_1^3 - 3x_1y_1^2\right) E + \left(3x_1^2y_1 - y_1^3\right) I
\end{aligned}
$$

4. (Quaternions) Consider the two complex 2x2 matrices

$$
A = \begin{pmatrix} z_1 & \zeta_1 \\ -\bar{\zeta}_1 & \bar{z}_1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} z_2 & \zeta_2 \\ -\bar{\zeta}_2 & \bar{z}_2 \end{pmatrix} .
$$

Show that the matrices

$$
C = A + B \qquad C = A\,B \qquad C = A\,B - B\,A \qquad C = A^{-1}
$$

have the same shape $C = \begin{pmatrix} z_3 & \zeta_3 \\ -\bar{\zeta}_3 & \bar{z}_3 \end{pmatrix}$ and give expressions for z_3 and ζ_3.

Let's consider the three quaternions

$$
I = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} , \; J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \; \text{and } K = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} .
$$

Show that $I^2 = J^2 = K^2 = IJK = -1$.

Solution:

(a)

$$
\begin{aligned}
A + B &= \begin{pmatrix} z_1 & \zeta_1 \\ -\bar{\zeta}_1 & \bar{z}_1 \end{pmatrix} + \begin{pmatrix} z_2 & \zeta_2 \\ -\bar{\zeta}_2 & \bar{z}_2 \end{pmatrix} \\
&= \begin{pmatrix} z_1 + z_2 & \zeta_1 + \zeta_2 \\ -\bar{\zeta}_1 - \bar{\zeta}_2 & \bar{z}_1 + \bar{z}_2 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
AB &= \begin{pmatrix} z_1 & \zeta_1 \\ -\bar{\zeta}_1 & \bar{z}_1 \end{pmatrix} \begin{pmatrix} z_2 & \zeta_2 \\ -\bar{\zeta}_2 & \bar{z}_2 \end{pmatrix} \\
&= \begin{pmatrix} z_1z_2 - \zeta_1\bar{\zeta}_2 & z_1\zeta_2 + \zeta_1\bar{z}_2 \\ -z_2\bar{\zeta}_1 - \bar{z}_1\bar{\zeta}_2 & -\bar{\zeta}_1\zeta_2 + \bar{z}_1\bar{z}_2 \end{pmatrix}
\end{aligned}
$$

$$
AB - BA = \begin{pmatrix} \zeta_2\bar{\zeta}_1 - \zeta_1\bar{\zeta}_2 & -\zeta_2\bar{z}_1 + \zeta_1\bar{z}_2 + \zeta_2 z_1 - \zeta_1 z_2 \\ z_1\zeta_2 - \bar{\zeta}_2\bar{z}_1 - z_2\bar{\zeta}_1 + \bar{\zeta}_1\bar{z}_2 & \zeta_1\bar{\zeta}_2 - \zeta_2\bar{\zeta}_1 \end{pmatrix}
$$

$$A^{-1} = \frac{1}{N}\begin{pmatrix} \bar{z}_1 & -\zeta_1 \\ \bar{\zeta}_1 & z_1 \end{pmatrix}$$

with $N = z_1\bar{z}_1 + \zeta_1\bar{\zeta}_1$. To verify this claim, we compute

$$\begin{aligned} AA^{-1} = \begin{pmatrix} z_1 & \zeta_1 \\ -\bar{\zeta}_1 & \bar{z}_1 \end{pmatrix} \frac{1}{N}\begin{pmatrix} \bar{z}_1 & -\zeta_1 \\ \bar{\zeta}_1 & z_1 \end{pmatrix} =& \frac{1}{N}\begin{pmatrix} z_1\bar{z}_1 + \zeta_1\bar{\zeta}_1 & -z_1\zeta_1 + \zeta_1 z_1 \\ -\bar{\zeta}_1\bar{z}_1 + \bar{z}_1\bar{\zeta}_1 & +\bar{\zeta}_1\zeta_1 + \bar{z}_1 z_1 \end{pmatrix} \\ =& \frac{1}{N}\begin{pmatrix} z_1\bar{z}_1 + \zeta_1\bar{\zeta}_1 & 0 \\ 0 & +\bar{\zeta}_1\zeta_1 + \bar{z}_1 z_1 \end{pmatrix} \\ =& \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

(b)

$$I^2 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} = \begin{pmatrix} i^2 & 0 \\ 0 & (-i)^2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$J^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$K^2 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{i}^2 & 0 \\ 0 & \mathrm{i}^2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{aligned} IJK =& (IJ)K = \left[\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right]\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \\ =& \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = K^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

Alternatively,

$$\begin{aligned} IJK =& I(JK) = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\right] \\ =& \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} = I^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

In passing, we note that

$$\begin{aligned} IJ =& K \\ JK =& I \\ KI =& \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = J \end{aligned}$$

Note that we keep the product IJ, JK, and KI in cyclic order. Otherwise, we get a sign change in the formulae above.