Statistical Models and Data Analysis

Summer term 2018

Problem Set 3 **23.4.2018**

The solutions to this exercise should be ready by 2 pm on 7.5.2018 . If you have any questions, please send me an email at stemmler@bio.lmu.de. You may submit your exercises by email in one file, but please be legible!

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1. (Complex numbers)

(a) Compute the real and imaginary part of the following expressions:

$$\mathrm{i} \qquad (3+\mathrm{i})-(5+2\,\mathrm{i}) \qquad (3+\mathrm{i})\,(5+2\,\mathrm{i}) \qquad (3+\mathrm{i})/(5+2\,\mathrm{i}) \qquad (3+\mathrm{i})^2 \qquad 4\,\mathrm{e}^{\mathrm{i}\pi/4}\ .$$

(b) Compute the modulus and argument of the following expressions:

$$\mathrm{i} \qquad 2\,\mathrm{e}^{\mathrm{i}\,\pi/2} \qquad (2+2\,\mathrm{i}) \qquad (\sqrt{3}-\mathrm{i}) \qquad \ln(-2) \qquad \sqrt{-2} \qquad (4\,\mathrm{e}^{\mathrm{i}\pi/4})^5\ .$$

(c) Find all complex numbers z that solve the equations

$$z^2 = \mathrm{i} \qquad z^3 - 1 = 0 \qquad z^4 + 2\,z^2 + 1 = 0\ .$$

2. (Complex functions) Consider $f(z) = z^2$ for $z = x + iy$. Let $u(x, y) = \mathrm{Re}\,[f(z)]$ be the real part and $v(x, y) = \mathrm{Im}\,[f(z)]$ be the imaginary part of $f(z)$. Show that, for this $f(z)$, the following hold:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\,\frac{\partial v}{\partial x}$$

The symbols $\partial/\partial x$ and $\partial/\partial y$ symbolize partial derivatives. For instance, $\dfrac{\partial u(x,y)}{\partial y}$ signifies taking the derivative of $u(x, y)$ with respect to y while keeping x constant.

3. (Complex numbers and matrices)

Consider the real matrices

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\ .$$

Show that for all matrices $A = x_1\,E + y_1\,I$ and $B = x_2\,E + y_2\,I$, the following matrices C can be expressed as $C = x_3\,E + y_3\,I$ and give the corresponding expressions for x_3 and y_3!

$$C = A + B \qquad C = A\,B \qquad C = B\,A \qquad C = A^{-1}\,B \qquad C = \exp[A]\,\exp[-B]$$

Note that the inverse of a 2x2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ equals

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

as long as $ad - bc \neq 0$.
The exponential of a matrix M is defined as

$$\exp[M] = \sum_{n=0}^{\infty} \frac{M^n}{n!}$$

where $n! = n(n-1)(n-2)\cdots 2 \cdot 1$, which is called the factorial of n.

Try to interpret these results in terms of complex numbers.

4. (Quaternions) Consider the two complex 2x2 matrices

$$A = \begin{pmatrix} z_1 & \zeta_1 \\ -\bar{\zeta}_1 & \bar{z}_1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} z_2 & \zeta_2 \\ -\bar{\zeta}_2 & \bar{z}_2 \end{pmatrix} .$$

Show that the matrices

$$C = A + B \qquad C = A\,B \qquad C = A\,B - B\,A \qquad C = A^{-1}$$

have the same shape $C = \begin{pmatrix} z_3 & \zeta_3 \\ -\bar{\zeta}_3 & \bar{z}_3 \end{pmatrix}$ and give expressions for z_3 and ζ_3.

Let's consider the three quaternions

$$I = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} , \; J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \text{ and } K = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} .$$

Show that $I^2 = J^2 = K^2 = IJK = -1$.