Statistical Models and Data Analysis

Summer term 2018

Problem Set 5 / Solutions 14.05.201

1. (Fourier transforms) Compute the Fourier transforms of the following functions:

$$g(x) = \exp\left(-\frac{x^2}{2}\right) \qquad \text{(use the fact that } \int_{-\infty}^{\infty} \exp(-x^2)\,dx = \sqrt{\pi})$$

$$\text{sinc}(x) = \frac{\sin(x)}{x} \qquad \text{(rewrite the sine function as the imaginary part of } \exp(ix)\text{)}$$

Plot both the original function and its Fourier transform.

Solution:

(a)

$$\begin{aligned}\widetilde{g}(\omega) &= \int_{-\infty}^{\infty} g(x)\exp(-\mathrm{i}\omega x)\ dx \\ &= \int_{-\infty}^{\infty} \exp\left(-x^2/2 - \mathrm{i}\omega x\right)\ dx\end{aligned}$$

Now complete the square in the argument to the exponential

$$\begin{aligned}\widetilde{g}(\omega) &= \int_{-\infty}^{\infty} \exp\left[-(x^2 - 2\mathrm{i}\omega x + (\mathrm{i}\omega)^2)/2 + (\mathrm{i}\omega)^2)/2\right]\ dx \\ &= \exp(-\omega^2/2)\int_{-\infty}^{\infty} \exp\left[-(x-\mathrm{i}\omega)^2/2\right]\ dx\end{aligned}$$

Substitute $z = (x + \mathrm{i}\omega)/\sqrt{2}$ and use the fact that $\int_{-\infty}^{\infty} \exp(-x^2)\,dx = \sqrt{\pi})$

$$\widetilde{g}(\omega) = \exp(-\omega^2/2)\sqrt{2\pi}$$

(b) Let us first analyze the Fourier transform of the derivative of a function

$$\mathcal{F}(f'(x)) = \int_{-\infty}^{\infty} \left(\frac{d}{dx}f(x)\right)\exp(-\mathrm{i}\omega x)\ dx$$

by integration by parts, and assuming that $f(x) \to 0$ as $x \to \pm\infty$

$$\begin{aligned}\mathcal{F}(f'(x)) &= -\int_{-\infty}^{\infty} f(x)\frac{d}{dx}\exp(-\mathrm{i}\omega x)\ dx \\ &= \mathrm{i}\omega\int_{-\infty}^{\infty} f(x)\exp(-\mathrm{i}\omega x)\ dx \\ &= \mathrm{i}\omega\mathcal{F}(f(x))\end{aligned}$$

We deduce that

$$\mathcal{F}\left(\int_{-\infty}^{x} f(z)dz\right) = \frac{1}{\mathrm{i}\omega}\mathcal{F}(f(x))$$

$$\begin{aligned}\widetilde{g}(\omega) &= \int_{-\infty}^{\infty} \frac{\sin(x)}{x}\exp\left(-\mathrm{i}\omega x\right)\ dx \\ &= \int_{-\infty}^{\infty} \left[\frac{\exp(\mathrm{i}x)}{2\mathrm{i}x} - \frac{\exp(-\mathrm{i}x)}{2\mathrm{i}x}\right]\exp\left(-\mathrm{i}\omega x\right)\ dx\end{aligned}$$

Substitute $x \to -x$

$$\begin{aligned}
\widetilde{g}(\omega) =& \int_{-\infty}^{\infty} \left[-\frac{\exp(-\mathrm{i}x)}{2\mathrm{i}x} + \frac{\exp(\mathrm{i}x)}{2\mathrm{i}x} \right] \exp(\mathrm{i}\omega x) \ dx \\
=& 2\pi \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[-\frac{\exp(-\mathrm{i}x)}{2\mathrm{i}x} + \frac{\exp(\mathrm{i}x)}{2\mathrm{i}x} \right] \exp(\mathrm{i}\omega x) \ dx \right\} \\
=& 2\pi \left\{ \mathcal{F}^{-1} \left(\frac{\exp(\mathrm{i}x)}{2\mathrm{i}x} \right) - \mathcal{F}^{-1} \left(\frac{\exp(-\mathrm{i}x)}{2\mathrm{i}x} \right) \right\}
\end{aligned}$$

Note

$$\begin{aligned}
\int_{-\infty}^{\infty} \delta(\omega - 1) \exp(-i\omega x) \ d\omega = \exp(-\mathrm{i}x) \\
\int_{-\infty}^{\infty} \delta(\omega + 1) \exp(-i\omega x) \ d\omega = \exp(\mathrm{i}x)
\end{aligned}$$

Hence,

$$\begin{aligned}
\widetilde{g}(\omega) =& 2\pi \left\{ \mathcal{F}^{-1} \left(\frac{\exp(\mathrm{i}x)}{2\mathrm{i}x} \right) - \mathcal{F}^{-1} \left(\frac{\exp(-\mathrm{i}x)}{2\mathrm{i}x} \right) \right\} \\
=& \pi \left\{ \mathcal{F}^{-1} \left(\frac{\mathcal{F}(\delta(\omega + 1)}{\mathrm{i}\omega} \right) - \mathcal{F}^{-1} \left(\frac{\mathcal{F}(\delta(\omega - 1)}{\mathrm{i}\omega} \right) \right\} \\
=& \pi \left\{ \mathcal{F}^{-1}\mathcal{F} \left(\int_{-\infty}^{\omega} \delta(s+1) - \delta(s-1) \ ds \right) \right\}
\end{aligned}$$

2. (Similarity transformation) Given a function $f(x)$, we can ask what happens to $f(x)$ and its Fourier transform $\mathcal{F}(f(x)) = \widetilde{f}(\omega)$ when we rescale the x-axis by letting $x \to ax$ with $a > 0$.

Using the definition of the Fourier transform, as shown in class, show that

$$\mathcal{F}(f(ax)) = \frac{1}{a} \widetilde{f} \left(\frac{s}{a} \right).$$

Use this property and the result from the first exercise to compute the Fourier transform of a Gaussian function $g(x) = \exp(-x^2/(2\sigma^2))$. If σ is large (and the Gaussian is broad), what happens to the Fourier transform?

Solution:

$$\mathcal{F}(f(x)) = \int_{-\infty}^{\infty} f(x) \exp(-\mathrm{i}sx) \ dx$$

So

$$\mathcal{F}(f(ax)) = \int_{-\infty}^{\infty} f(ax) \exp(-\mathrm{i}sx) \ dx$$

With variable substitution $y = ax$ (and hence $dx = dy/a$) we get

$$\begin{aligned}
\mathcal{F}(f(ax)) =& \frac{1}{a} \int_{-\infty}^{\infty} f(y) \exp(-\mathrm{i}s/ay) \ dx \\
=& \frac{1}{a} \widetilde{f}(s/a)
\end{aligned}$$

Applying this to the result from the first exercise, we get

$$\mathcal{F}\left[\exp(-x^2/(2\sigma^2))\right] = \sqrt{2\pi\sigma^2}\exp\left(-\frac{s^2\sigma^2}{2}\right)$$

3. (Fourier transform of an amplitude modulated signal)

Consider the function

$$f(t) = [1 + a\,\sin(\omega_m\,t)]\,\sin(\omega_c\,t)$$

with $0 \leq a \leq 1$.

Compute the Fourier-Transform of $f(t)$. To do so, express the cosine functions in terms of complex exponentials and use the formula for the Fourier representation of the delta function

$$\delta(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}xy}\mathrm{d}x$$

. **Solution:**

$$\begin{aligned} f(t) =& [1 + a\,\sin(\omega_m\,t)]\,\sin(\omega_c\,t) \\ =& \left[1 + \frac{a}{2\mathrm{i}}\left(\exp(\mathrm{i}\omega_m t) - (\exp(-\mathrm{i}\omega_m t)\right)\right]\frac{(\exp(\mathrm{i}\omega_c t) - (\exp(-\mathrm{i}\omega_c t))}{2\mathrm{i}} \end{aligned}$$

The Fourier transform is

$$\mathcal{F}(f) = \int_{-\infty}^{\infty} f(t)\exp(-\mathrm{i}\omega t)\,dt$$

We are going to use

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t = 2\pi\delta(\omega).$$

Moreover, $\delta(\omega) = \delta(-\omega)$. Returning to the function $f(t) = [1 + a\,\sin(\omega_m\,t)]\,\sin(\omega_c\,t)$, we see that if we plug this into the definition of the Fourier transform, we will get exponential terms with arguments

$$\pm\omega_c - \omega,\ \pm\omega_m \pm \omega_c - \omega$$

Keeping the coefficients straight, the end result is

$$\frac{\pi}{2}a\left[\delta(\omega + \omega_c - \omega_m) + \delta(\omega - \omega_c + \omega_m) - \delta(-\omega + \omega_c + \omega_m) - \delta(\omega + \omega_c + \omega_m)\right] - i\pi\delta(\omega_c - \omega) + i\pi\delta(\omega + \omega_c)$$